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Hamiltonian dynamics and gauge symmetry for three-dimensional Palatini theory with cosmological constant

Alberto Escalante^{a,1} and Omar Rodríguez Tzompantzi^b

^a*Instituto de Física, Benemérita Universidad Autónoma de Puebla,
Apartado Postal J-48 72570, Puebla Pue., México*

^b*Facultad de Ciencias Físico Matemáticas, Benemérita Universidad Autónoma de Puebla,
Apartado postal 1152, 72001 Puebla, Pue., México*

E-mail: aescalan@ifuap.buap.mx, omar.tz2701@gmail.com

ABSTRACT: A pure Dirac's framework for 3D Palatini's theory with cosmological constant is performed. By considering the complete phase space, we find out the full structure of the constraints, and their corresponding algebra is computed explicitly. We report that in order to obtain a well defined algebra among the constraints, the internal group corresponds to $SO(2, 1)$. In addition, we obtain the extended action, the extended Hamiltonian, the gauge symmetry, and the Dirac brackets of the theory. Finally, we compare our results with those reported in the literature.

KEYWORDS: Classical Theories of Gravity, Topological Field Theories

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¹Corresponding author.

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1 Introduction

Models describing 3D gravity have been used as an alternative tool in order to clarify the highly complex dynamical behavior of the realistic four-dimensional general relativity [GR]. A well-known theory describing gravity in 3D is the so-called Palatini's theory, where the connection and the triad fields are the fundamental dynamical variables. This feature has been of great interest in the community for the construction of a non-perturbative quantum gravity and its cosmological applications [1–7]. In spite that the Hamiltonian formalism of 3D Palatini's theory has been studied in so many works with or without cosmological constant Λ [5, 8–10], there are still difficulties to find out the correct symmetries of the theory, namely, the gauge symmetry and the internal group. In this respect, some authors have claimed that the gauge symmetry of 3D [GR] is Poincaré symmetry [8], whereas other ones have claimed that it is the Lorentz symmetry plus diffeomorphisms [10]. On the other hand, with respect to the internal group, in [9] it was established that the internal group is $\text{ISO}(2, 1)$ Poincaré, whereas in [5] it is reported that 3D Palatini's theory with or without a cosmological constant Λ is well-defined for a wide class of Lie groups; if $\Lambda = 0$, then the internal group \mathcal{G} can be arbitrary, but if $\Lambda \neq 0$, then the internal group \mathcal{G} has to admit an invariant totally anti-symmetric tensor ϵ^{IJK} . Moreover, if GR is coupled with matter fields, then the internal group \mathcal{G} corresponds to $\text{SO}(2, 1)$.

It is well-known that the concept of gauge symmetry plays an important role in modern physics; the physics of the fundamental interactions based on the standard model [11], is a relevant example where the symmetries of a dynamical system just like gauge covariance is useful for understanding the classical and quantum formulation of the theory. In this sense, the canonical study of GR is an important step to achieve in order to construct a background independent quantum theory. Thus, the canonical framework for singular

theories is the best tool that we have at hand for studying these relevant symmetries, just as gauge invariance. It is clear that in any theory presenting a kind of symmetries not all of them correspond to gauge symmetries; in fact, the gauge symmetry of any theory is obtained by following all Dirac's steps [12–17], and hence we need to construct, according to Dirac's conjecture, a gauge generator by using the first class constraints. It is worth mentioning that usually Dirac's formalism is not carried out properly, namely, a complete Hamiltonian formulation means that all steps of the Dirac procedure should be performed, and if some of these steps are missing or implemented incorrectly, then we cannot be sure that the correct analysis has been carry out, and it is possible that the symmetries found are not correct. In fact, usually the people prefers to work on a smaller phase space context (also called standard approach); this means that only those variables that occur in the action with temporal derivative are considered as dynamical, however, this approach is applicable only when the theory presents certain simplicity; in general in order to obtain a complete description one must perform a complete analysis by following all Dirac's steps [18–23].

The aim of the present paper is to perform a pure Dirac's analysis for 3D Palatini's theory with cosmological constant. Thus, in our analysis we shall consider all the set of one-forms " A^{IJ} " and " e^I " that define our theory as dynamical ones and not only those variables that occur with time derivatives in the Lagrangian density, as it is done usually in the standard approach (See ref. [5, 8] for a standard analysis). The price to pay by working with a smaller configuration space, is that we are not able to know the complete form of the constraints, neither the full form of the gauge transformations defined on the complete phase space nor the complete algebra among the constraints for the theory. Of course, by working with the full configuration space we can obtain a best and complete description of the theory at a classical level. By working with a complete phase space, we obtain the full structure of the constraints and their full algebra; throughout this paper, we show that the Poisson algebra among the constraints is closed provided that the internal group corresponds to $SO(2, 1)$. We also show that if we take the cosmological constant as zero, then the Poisson algebra among the constraints is closed provided that the internal group is still $SO(2, 1)$. Hence, it is not necessary to couple matter fields to gravity in order to conclude that for GR in 3D the internal group corresponds to $SO(2, 1)$, as it is reported in [5]; in fact, we obtain the same conclusion by performing a detailed canonical analysis without adding matter fields. It is important to comment that this result has not been reported in the literature because it is common that some of Dirac's steps are omitted. In this respect, we can observe in [9] a canonical analysis of Palatin's theory without cosmological constant, and that work reported that the algebra among the first class constraints form an $ISO(2, 1)$ algebra, and therefore, the gauge symmetry is Poincaré symmetry; however, in that work the second class constraints were solved and the Dirac brackets were not constructed; we think that those results are incomplete. In this paper we obtain the full structure of the first class constraints and the gauge symmetry for 3D Palatini's theory with or without cosmological constant; we also construct the fundamental Dirac's brackets and by using these brackets, we compute Dirac's algebra among the constraints showing that the algebra is closed. In addition, we compare our results with those found in the literature.

2 A pure Dirac's analysis

It is well known that Palatini's action with cosmological constant can be written as

$$S[A, e]_P = \int_{\mathcal{M}} \epsilon^{IJK} \left[R[A]_{IJ} \wedge e_K - \frac{\Lambda}{3} e_I \wedge e_J \wedge e_K \right], \quad (2.1)$$

where $A^{IJ} = A_\mu^{IJ} dx^\mu$ is a connection 1-form valued on any Lie group \mathcal{G} that admits an invariant, totally anti-symmetric tensor ϵ^{IJK} [5]; $e^I = e_\mu^I dx^\mu$ is a triad 1-form that represents the field gravitational, Λ is the cosmological constant, and R^{IJ} is the curvature 2-form of the connection A^{IJ} , *i.e.*, $R^{IJ} \equiv dA^{IJ} + A_L^I \wedge A^{LJ}$. Here, x^μ are the coordinates that label the points of the 3-dimensional manifold \mathcal{M} . In our notation, Greek letters run from 0 to 2, while Latin letters will run from 0 to $\mathfrak{g} = \dim(\mathcal{G})$. From now on, we will take into account in the number of dynamical variables, constraints etc., only the space-time indices, this fact does not affect our results; at the end of our calculations, we will take into account the number of generators of the group.

The equations of motion that arise from the variation of the action (2.1) with respect to the dynamical variables are given by

$$\epsilon^{\alpha\mu\nu} \epsilon^{IJK} D_\mu e_{\nu K} = 0, \quad (2.2)$$

$$\epsilon^{\alpha\mu\nu} \epsilon_{IJK} [R_{\mu\nu}^{JK} - \lambda e_\mu^J e_\nu^K] = 0, \quad (2.3)$$

where (2.2) is the zero-torsion condition, which can be solved to get the unique torsion-free spin-connection compatible with e_α^I . Inserting the solution of (2.2) into (2.3), one gets Einstein's equation

$$G^{\mu\nu} + \Lambda g^{\mu\nu} = 0. \quad (2.4)$$

We recall that this equation implies that the space-time has constant scalar curvature proportional to Λ , *i.e.*, $R = 6\Lambda$. Note that this result is independent of the signature of the space-time and therefore, of the internal group that we are considering. Moreover, in order to perform the Hamiltonian analysis, we will assume that the manifold \mathcal{M} is topologically $\Sigma \times \mathcal{R}$, where Σ corresponds to a Cauchy's surface without boundary ($\partial\Sigma = 0$) and \mathcal{R} represents an evolution parameter.

By performing the 2 + 1 decomposition of our fields without breaking the internal \mathcal{G} symmetry and also without fixing any gauge, we can write the action (2.1) as

$$S[e, A]_P = \int dx^3 \left[\epsilon^{0ab} e_b^K \epsilon_{IJK} \dot{A}_a^{IJ} - \epsilon^{0ab} e_b^K \epsilon_{IJK} D_a A_0^{IJ} + \frac{1}{2} \epsilon^{0ab} \epsilon_{IJK} e_0^K F_{ab}^{IJ} - \Lambda \epsilon^{0ab} \epsilon_{IJK} e_0^I e_a^J e_b^K \right], \quad (2.5)$$

where $D_a A_b^{IJ} = \partial_a A_b^{IJ} + A_a^{IK} A_{bK}^J + A_a^{JK} A_{bK}^I$ and $F_{ab}^{IJ} = \partial_a A_b^{IJ} - \partial_b A_a^{IJ} + A_a^{IK} A_{bK}^J - A_b^{IK} A_{aK}^J$. Here $a, b = 1, 2$ are space coordinate indices. From (2.5) we can identify the following Lagrangian density

$$\mathcal{L} = \epsilon^{0ab} e_b^K \epsilon_{IJK} \dot{A}_a^{IJ} - \epsilon^{0ab} e_b^K \epsilon_{IJK} D_a A_0^{IJ} + \frac{1}{2} \epsilon^{0ab} \epsilon_{IJK} e_0^K F_{ab}^{IJ} - \Lambda \epsilon^{0ab} \epsilon_{IJK} e_0^I e_a^J e_b^K. \quad (2.6)$$

We have commented above that usually the Hamiltonian analysis of (2.1) is carried out in a reduced phase space [5, 8]; this means that in those works the 1-forms A^{IJ} and e^I that occur in the action with time derivative are considered as dynamical variables. However, in this paper we will not work in that form, we shall perform our analysis in concordance with the background independence of the theory, this means, we shall consider all A^{IJ} and e^I as our set of dynamical variables which define our theory. Hence, by identifying our set of dynamical variables, a pure Dirac's method requires to define the momenta $(\Pi^\alpha_I, \Pi^{\alpha\beta}_{IJ})$ canonically conjugated to $(e_\alpha^I, A_\alpha^{IJ})$ [12, 13]

$$\Pi^\alpha_{IJ} = \frac{\delta \mathcal{L}}{\delta \dot{A}_\alpha^{IJ}}, \quad \Pi^\alpha_I = \frac{\delta \mathcal{L}}{\delta \dot{e}_\alpha^I}. \quad (2.7)$$

On the other hand, the matrix elements of the Hessian

$$\frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu e_\alpha^I) \partial(\partial_\mu e_\beta^I)}, \quad \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu e_\alpha^I) \partial(\partial_\mu A_\beta^{IJ})}, \quad \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu A_\alpha^{IJ}) \partial(\partial_\mu A_\beta^{IJ})},$$

are identically zero and the rank of the matrix Hessian is zero. Thus, we expect 6 primary constraints. From the definition of the momenta (2.7), we identify the following primary constraints

$$\begin{aligned} \phi^0_I &:= \Pi^0_I \approx 0, & \phi^a_I &:= \Pi^a_I \approx 0, \\ \phi^0_{IJ} &:= \Pi^0_{IJ} \approx 0, & \phi^a_{IJ} &:= \Pi^a_{IJ} - \epsilon^{0ab} \epsilon_{IJK} e_b^K \approx 0. \end{aligned} \quad (2.8)$$

We can observe that, if a smaller phase space is considered, the ϕ^0_I and ϕ^0_{IJ} constraints would not be taken into account [5, 8]. However, the purpose of this paper is to work with the complete phase space and so they are crucial for our study. The canonical Hamiltonian of the theory is given by

$$H_C = \int \left[-\frac{1}{2} e_0^K \epsilon^{0ab} \epsilon_{IJK} F^{IJ}_{ab} - A_0^{IJ} D_a \Pi^a_{IJ} + \Lambda e_0^I e_a^J \Pi^a_{IJ} \right] dx^2. \quad (2.9)$$

In this manner, the primary Hamiltonian will be constructed by adding the primary constraints (2.8) to (2.9), namely

$$H_P = H_C + \int [\lambda_0^I \phi^0_I + \lambda_a^I \phi^a_I + \lambda_0^{IJ} \phi^0_{IJ} + \lambda_a^{IJ} \phi^a_{IJ}] dx^2, \quad (2.10)$$

here $\lambda^I_0, \lambda^I_a, \lambda^{IJ}_0, \lambda^{IJ}_a$ are Lagrange multipliers enforcing the constraints. In this theory, the non-vanishing fundamental Poisson brackets are given by

$$\begin{aligned} \{e_\alpha^I(x), \Pi^\beta_J(y)\} &= \delta^\beta_\alpha \delta^I_J \delta^2(x-y), \\ \{A_\alpha^{IJ}(x), \Pi^\beta_{KL}(y)\} &= \frac{1}{2} \delta^\beta_\alpha (\delta^I_K \delta^J_L - \delta^I_L \delta^J_K) \delta^2(x-y). \end{aligned} \quad (2.11)$$

Now, it is necessary to identify if the theory have secondary constraints. For this aim, we observe that the (6×6) matrix whose entries are the Poisson brackets of the primary constraints (2.8), has rank = 4 and 2 null vectors. Therefore, from the consistency conditions and the null vectors we get the following 2 secondary constraints

$$\begin{aligned} \dot{\phi}^0_I &= \{\phi^0_I(x), H_P\} \approx 0 \implies \psi_I := -\frac{1}{2} \epsilon^{0ab} \epsilon_{IKL} F_{ab}^{KL} + \Lambda e_a^J \Pi^a_{IJ} \approx 0, \\ \dot{\phi}^0_{IJ} &= \{\phi^0_{IJ}(x), H_P\} \approx 0 \implies \psi_{IJ} := D_a \Pi^a_{IJ} \approx 0. \end{aligned} \quad (2.12)$$

The rank allows us to get the following expressions for the Lagrange multipliers

$$\begin{aligned}\dot{\phi}^a_{IJ} = \{\phi^a_{IJ}(x), H_P\} \approx 0 &\implies \epsilon^{0ab}\epsilon_{IJK}\lambda_b^K = 2\epsilon^{0ab}\epsilon_{IJK}D_b e_0^K + A_{0I}{}^K \Pi^a_{JK} - A_{0J}{}^K \Pi^a_{IK}, \\ \dot{\phi}^a_I = \{\phi^a_I(x), H_P\} \approx 0 &\implies \epsilon^{0ab}\epsilon_{IJK}\lambda_b^{JK} = \Lambda \Pi^a_{IJ} e_0^J.\end{aligned}\quad (2.13)$$

In this theory there are not third constraints. At this point, we need to identify the first- and second-class constraints from the primary and secondary ones. In order to achieve this aim, it is necessary to calculate the $[8 \times 8]$ matrix whose entries are the Poisson brackets constructed out of the primary and secondary constraints. The non-vanishing Poisson brackets are given by

$$\begin{aligned}\{\phi_I^a(x), \phi_{KL}^b(y)\} &= -\epsilon^{0ab}\epsilon_{IKL}\delta^2(x-y), \\ \{\phi_{IJ}^a(x), \psi_K(y)\} &= \epsilon^{0ab}[-\epsilon_{KIJ}\partial_b + \epsilon_{KIL}A_{bJ}{}^L + \epsilon_{KLJ}A_{bI}{}^L]\delta^2(x-y), \\ \{\phi_{IJ}^a(x), \psi_{KL}(y)\} &= \frac{1}{2}[\eta_{LJ}\Pi_{KI}^a - \eta_{LI}\Pi_{KJ}^a + \eta_{KJ}\Pi_{IL}^a - \eta_{KI}\Pi_{JL}^a]\delta^2(x-y), \\ \{\psi_I(x), \psi_J(y)\} &= -\Lambda\epsilon^{0ab}\epsilon_{IJH}D_a e_b^H\delta^2(x-y), \\ \{\psi_I(x), \psi_{KL}(y)\} &= \frac{1}{2}[\eta_{KI}\psi_L - \eta_{LI}\psi_K + \Lambda(e_{Ka}\Pi_{IL}^a - e_{La}\Pi_{IK}^a)]\delta^2(x-y), \\ \{\psi_{IJ}(x), \psi_{KL}(y)\} &= \frac{1}{2}[\eta_{KI}\psi_{LJ} - \eta_{LI}\psi_{KJ} + \eta_{KJ}\psi_{IL} - \eta_{JL}\psi_{IK}]\delta^2(x-y) \approx 0.\end{aligned}\quad (2.14)$$

This matrix has rank 4 and 4 null-vectors. Thus, we expect 4 second-class constraints and 4 first-class constraints respectively. From the null vectors we can identify the following complete structure of the first-class constraints

$$\begin{aligned}\gamma^0_I &:= \Pi^0_I \approx 0, & \gamma_I &:= -D_a\phi^a_I - \frac{1}{2}\epsilon^{0ab}\epsilon_{IKL}F_{ab}{}^{KL} + \Lambda e_a^J \Pi^a_{IJ} + \Lambda e_a^J \phi^a_{IJ} \approx 0, \\ \gamma^0_{IJ} &:= \Pi^0_{IJ} \approx 0, & \gamma_{IJ} &:= D_a\Pi^a_{IJ} + \frac{1}{2}\epsilon^H{}_{IM}\epsilon^{MF}{}_J[\phi^a_F e_{aH} - \phi^a_H e_{aF}] \approx 0.\end{aligned}\quad (2.15)$$

We can observe that the γ_I constraint can be thought as the dynamical constraint whereas the γ_{IJ} constraint can be identified with the Gauss constraint for the theory as occurs in Yang-Mills theory. On the other hand, the rank of the matrix (2.14) yields the following second-class constraints

$$\phi^a_I : \chi^a_I = \Pi^a_I \approx 0, \quad \phi^a_{IJ} : \chi^a_{IJ} = \Pi^a_{IJ} - \epsilon^{0ab}\epsilon_{IJK}e_b^K \approx 0. \quad (2.16)$$

It is important to remark that the complete structure of the constraints γ_I and γ_{IJ} given in (2.15) are fixed through the null vectors and they are of first-class. In this way, the method itself allows us to find by means of the rank and the null vectors of the matrix (2.14) all the complete structure of the first- and second-class constraints [14–21]. This is the advantage of a pure Dirac method when it is applied without missing any step. It is worth mentioning, that in the complete structure of the first class constraints occur the second class constraints, and this full structure has not been reported in the literature. In fact, we are able to observe that our constraints and the constraints reported in [5, 8, 9] are not the same. On one hand, in [5, 8] they work in a smaller phase space context. On the other hand, in [9] they solved the second class constraints before performing the contraction of

the primary and secondary constraints with the null vectors, hence, our constraints are different from those reported in [9].

Now, we will observe the implications obtained by working with a pure Dirac's analysis; the non zero algebra among all the constraints (2.15) and (2.16) is given by

$$\{\gamma_I(x), \gamma_J(y)\} = 2\Lambda\gamma_{JI}\delta^2(x-y) \approx 0, \quad (2.17)$$

$$\{\gamma_I(x), \gamma_{KL}(y)\} = \frac{1}{2}[\eta_{IK}\gamma_L - \eta_{IL}\gamma_K]\delta^2(x-y) \approx 0, \quad (2.18)$$

$$\{\gamma_{IJ}(x), \gamma_{KL}(y)\} = \frac{1}{2}[\eta_{IK}\gamma_{LJ} - \eta_{IL}\gamma_{KJ} + \eta_{JK}\gamma_{IL} - \eta_{JL}\gamma_{IK}]\delta^2(x-y) \approx 0, \quad (2.19)$$

$$\{\gamma_I(x), \chi^a_J(y)\} = 2\Lambda\chi^a_{IJ}\delta^2(x-y) \approx 0, \quad (2.20)$$

$$\{\gamma_I(x), \chi^a_{KL}(y)\} = \frac{1}{2}[\eta_{IL}\chi^a_K - \eta_{IK}\chi^a_L]\delta^2(x-y) \approx 0, \quad (2.21)$$

$$\{\gamma_{IJ}(x), \chi^a_K(y)\} = \frac{1}{2}[\eta_{KJ}\chi^a_I - \eta_{KI}\chi^a_J]\delta^2(x-y) \approx 0, \quad (2.22)$$

$$\{\gamma_{IJ}(x), \chi^a_{KL}(y)\} = \frac{1}{2}[\eta_{KJ}\chi^a_{IL} - \eta_{KI}\chi^a_{JL} + \eta_{LI}\chi^a_{JK} - \eta_{LJ}\chi^a_{IK}]\delta^2(x-y) \approx 0, \quad (2.23)$$

$$\{\chi^a_I(x), \chi^b_{KL}(y)\} = -\epsilon^{0ab}\epsilon_{IKL}\delta^2(x-y), \quad (2.24)$$

Therefore, we can observe from (2.19) and (2.23) (see the appendix A for details) that the algebra is closed and has a desired form provided that

$$\epsilon^{IJK}\epsilon_{IMN} = (-1)(\delta^J_M\delta^K_N - \delta^J_N\delta^K_M). \quad (2.25)$$

This is a property of the structure constants $\epsilon_{JK}^I = \epsilon^{IMN}\eta_{MJ}\eta_{NK}$ of the Lie algebra of $\text{SO}(2,1)$. Thus, the constraints (2.15) and (2.16) are closed under Poisson brackets, i.e., they form first- and second-class constraint sets respectively provided that $\mathcal{G} = \text{SO}(2,1)$.

Furthermore, we can observe that if we take $\Lambda \rightarrow 0$, the algebra of constraints (2.15) form a Poincaré algebra as that reported in [9], but the internal group is still $\text{SO}(2,1)$ as can be observed from the algebra of the constraints (2.19) and (2.23) (see the appendix A). We appreciate a difference among the analysis performed in [9] and the analysis of this work. In fact, in that work there are not second class constraints and they do not construct the Dirac's brackets. In this paper, we will preserve the second class constraints until the end of the calculations. At the end of our analysis, we construct the Dirac's brackets as is shown below. We can also observe that if we use the adjoint representation of the Lie algebra of the group \mathcal{G} , then it is not necessary to restrict ourselves to $\text{SO}(2,1)$ and the algebra among the constraints do not form a Poincaré algebra (see appendix C), this a clear difference among our results and those reported in the literature.

The correct identification of first- and second-class constraints allows us to carry out the counting of degrees of freedom as follows; there are 12 canonical variables ($e_\alpha^I, \Pi^\alpha_I, A_\alpha^{IJ}, \Pi^\alpha_{IJ}$), 4 independent first-class constraints ($\gamma^0_I, \gamma^0_{IJ}, \gamma_I, \gamma_{IJ}$), and 4 independent second-class constraints (χ^a_I, χ^a_{IJ}) (we have taken into account only the space-time indices). Thus, we conclude that 3D Palatini's theory with cosmological constant lacks of physical degrees of freedom, i.e., it defines a topological field theory as expected. On the other hand, in order to obtain the extended action and the fundamental Dirac's brackets,

we need to determinate the unknown Lagrange multipliers. For this aim, we introduce the matrix $C_{\alpha\beta}$ whose elements are the Poisson brackets of the second-class constraints given by

$$C_{\alpha\beta} = \begin{pmatrix} 0 & -\epsilon^{0ab}\epsilon_{IJK}\delta^2(x-y) \\ \epsilon^{0ab}\epsilon_{IJK}\delta^2(x-y) & 0 \end{pmatrix}, \quad (2.26)$$

the Dirac's bracket of two functionals A, B defined on the phase space, is expressed by

$$\{F(x), G(y)\}_D \equiv \{F(x), G(y)\} + \int d^2z d^2w \{F(x), \xi^\alpha(z)\} C^{-1}_{\alpha\beta} \{\xi^\alpha(w), G(y)\},$$

where $\{F(x), G(y)\}$ is the Poisson bracket between two functionals F, G , and $\xi^\alpha(z) = (\chi_I^a, \chi_{IJ}^a)$ are the second-class constraints, and $C_{\alpha\beta}^{-1}$ is the inverse of (2.26) that has a trivial form. For simplicity we will restrict the Dirac algebra to the particular case of $\mathcal{G} = \text{SO}(2, 1)$, by using this fact, Dirac's brackets of the dynamical variables are given by

$$\begin{aligned} \{e_a^I(x), \Pi_J^b(y)\}_D &= 0, \\ \{e_a^I(x), A_b^{KL}(y)\}_D &= -\frac{1}{2}\epsilon_{0ab}\epsilon^{IKL}\delta^2(x-y), \\ \{A_b^{KL}(x), e_a^I(y)\}_D &= \frac{1}{2}\epsilon_{0ab}\epsilon^{IKL}\delta^2(x-y), \\ \{A_a^{IJ}(x), \Pi_{KL}^b(y)\}_D &= \frac{1}{2}(\delta_K^I\delta_L^J - \delta_L^I\delta_K^J)\delta_a^b\delta^2(x-y). \end{aligned} \quad (2.27)$$

It is well known that the Dirac brackets (2.27) are essential ingredients in the quantization of the theory [1–4]. By using the brackets given in (2.27), we also compute the Dirac brackets among the constraints, we show that the Dirac algebra is closed (see appendix B). For the case of general internal groups, we will report the closure of Dirac's brackets in forthcoming works. Furthermore, the number of second class constraints fixes the number of indeterminate Lagrange multipliers, thus, by using the fact that the internal group for the theory is $\text{SO}(2, 1)$ and from (2.13) we identify the following Lagrange multipliers,

$$\lambda_a^{IJ} := \frac{\Lambda}{4}\epsilon_{0ab}\epsilon^{IJK}\Pi_{KL}^b e_0^L, \quad \lambda_a^I := D_a e_0^I + \frac{1}{2}\epsilon_{0ab}\epsilon^{IJK}A_{0J}^L \Pi_{KL}^b. \quad (2.28)$$

Hence, we use the Lagrange multipliers (2.28), the first-class constraints (2.15), and the second-class constraints (2.16) in order to identify the extended action for the theory

$$\begin{aligned} S_E = \int d^3x \Big[& \Pi_{IJ}^0 \dot{A}_0^{IJ} + \Pi_{IJ}^a \dot{A}_a^{IJ} + \Pi_{I\dot{0}}^0 \dot{e}_0^I - \Pi_{I\dot{a}}^a \dot{e}_a^I - \mathcal{H} - \zeta_0^I \gamma^0_I - \zeta_0^{IJ} \gamma^0_{IJ} \\ & - \zeta^I \gamma_I - \zeta^{IJ} \gamma_{IJ} - \bar{\lambda}_a^I \chi^a_I - \bar{\lambda}_a^{IJ} \chi^a_{IJ} \Big], \end{aligned} \quad (2.29)$$

where \mathcal{H} is a combination of first-class constraints

$$\mathcal{H} = e_0^I \gamma_I - A_0^{IJ} \gamma_{IJ} \approx 0, \quad (2.30)$$

and $\zeta_0^I, \zeta_0^{IJ}, \zeta^I, \zeta^{IJ}, \bar{\lambda}_a^I, \bar{\lambda}_a^{IJ}$, are the Lagrange multipliers that enforce the first- and second-class constraints. From the extended action we can identify the extended Hamiltonian, which is given by

$$H_E = \int d^2x \Big[\mathcal{H} + \zeta_0^I \gamma^0_I + \zeta_0^{IJ} \gamma^0_{IJ} + \zeta^I \gamma_I + \zeta^{IJ} \gamma_{IJ} \Big], \quad (2.31)$$

thus, the extended Hamiltonian is a linear combination of first-class constraints as expected. It is important to comment, that an extended Hamiltonian with the same structure than (2.30) is reported in [9], however, we need take into account that our first class constraints have a complete structure.

On the other hand, it is well-know that in GR the dynamical evolution is governed by the constraints reflecting the general covariance of the theory. Moreover, in order to perform the quantization of the theory, it is not possible to construct the Schrödinger equation because the action of the Hamiltonian on physical states is annihilated. The quantization process is carried out by the implementation of Dirac's quantization program for gauge systems with general covariance as that realized in Loop Quantum Gravity [1–4], or as it is showed in [14]; the first class constraints are promoted to operators \hat{C}_i on the kinematical Hilbert space and the physical states are those for which the Dirac conditions $\hat{C}_i \cdot \Psi = 0$, are satisfied. Hence, it is mandatory to perform a pure Dirac's analysis in order to identify the complete structure of the constraints, because constraints are the best guideline to perform the quantization.

We finish this paper by finding out the gauge symmetry. In fact, one of the most important symmetries present in singular theories with first class constraints is the gauge symmetry, because it can help us to identify physical observables [13]. Thus, we need to know explicitly the fundamental gauge transformations for the theory. For this aim, we will apply Castellani's algorithm [24] to construct the gauge generator. We define the generator of gauge symmetry as

$$G = \int d^2x \left[D_0 \varepsilon^I \gamma^0_I + \varepsilon^I \gamma_I + D_0 \kappa_0^{IJ} \gamma^0_{IJ} + \kappa^{IJ} \gamma_{IJ} \right], \quad (2.32)$$

Thus, we find that the gauge symmetry on the phase space are given by

$$\begin{aligned} \delta e_a^I &= D_a \varepsilon^I + \kappa^{IJ} e_{aJ}, \\ \delta e_0^I &= D_0 \varepsilon_0^I, \\ \delta A_a^{IJ} &= D_a \kappa^{IJ} + \Lambda \varepsilon^I e_a^J - \Lambda \varepsilon^J e_a^I, \\ \delta A_0^{IJ} &= D_0 \kappa_0^{IJ}, \\ \delta \Pi^a_I &= 2\Lambda (\Pi^a_{IJ} - \epsilon^{0ab} \epsilon_{IJK} e_b^K) \varepsilon^J + \kappa_{IJ} \Pi^{aJ}, \\ \delta \Pi^0_I &= 0, \\ \delta \Pi^a_{IJ} &= \epsilon^{0ab} \epsilon_{IJM} D_b \varepsilon^M + \frac{1}{2} (\varepsilon_I \Pi^a_J - \varepsilon_J \Pi^a_I) + \kappa_I^N \Pi^a_{NJ} - \kappa_J^N \Pi^a_{NI}, \\ \delta \Pi^0_{IJ} &= 0. \end{aligned} \quad (2.33)$$

However, they can be written in covariant form by choosing the parameter in the following form; $\varepsilon^I = \varepsilon_0^I = \Theta^I$, $\kappa_0^{IJ} = -\kappa^{IJ} = \Delta^{IJ}$, thus, we get the following gauge symmetry for this theory

$$\begin{aligned} e_\mu^I &\longrightarrow e_\mu^I + D_\mu \Theta^I + \Delta^{IJ} e_{J\mu}, \\ A_\mu^{IJ} &\longrightarrow A_\mu^{IJ} + D_\mu \Delta^{IJ} + \Lambda \Theta^I e_\mu^J - \Lambda \Theta^J e_\mu^I. \end{aligned} \quad (2.34)$$

It is important to remark that (2.34) correspond to the gauge symmetry of the theory but they do not correspond to diffeomorphisms, instead they are an Λ -deformed Poincaré transformations (see appendix D). Nevertheless, we can redefine the gauge parameters as

$$\Theta^I = \frac{1}{2}\xi^\alpha e_\alpha^I \quad \text{and} \quad \Delta^{IJ} = \frac{1}{2}\xi^\alpha A_\alpha^{IJ}. \quad (2.35)$$

In this manner from the gauge symmetry we obtain

$$\begin{aligned} e_\mu^I &\longrightarrow e_\mu^I + \mathcal{L}_\xi e_\mu^I + \frac{1}{2}\xi^\alpha [D_\mu e_\alpha^I - D_\alpha e_\mu^I], \\ A_\mu^{IJ} &\longrightarrow A_\mu^{IJ} + \mathcal{L}_\xi A_\mu^{IJ} + \xi^\alpha \left[R_{\mu\alpha}^{IJ} - \frac{\Lambda}{2} (e_\mu^I e_\alpha^J - e_\alpha^I e_\mu^J) \right], \end{aligned} \quad (2.36)$$

which are (*on-shell*) diffeomorphisms, and this symmetry is contained in the gauge symmetry (2.34). We observe in (2.34) that if $\Lambda = 0$ we obtain as gauge symmetry the Poincaré transformations. Thus, we conclude that for GR without cosmological constant Poisson algebra is closed if the internal group $\text{SO}(2, 1)$ and the gauge symmetries are Poincaré transformations, extending the results reported in [9]. Furthermore, for GR with cosmological constant Poisson algebra close if the internal group is $\text{SO}(2, 1)$ and the gauge symmetries are a Λ -deformed Poincaré transformations. The Dirac algebra performed for a general internal group is not yet solved, we are working on this subject, which will be reported in elsewhere.

3 Summary and conclusions

In this paper, we have performed a pure Hamiltonian analysis for Palatini's theory with a cosmological constant. In order to obtain the best description of this theory, all the steps of Dirac's framework on full configuration space were followed. By means of the null vectors and the rank of the matrix whose elements are the Poisson brackets among primary and secondary constraints, we can identify the complete structure of the first and second class constraints. With the complete structure of the constraints and their algebra, we conclude that the Poisson algebra of 3D Palatini theory with a cosmological constant Λ is well-defined provided the internal group \mathcal{G} is $\text{SO}(2, 1)$. Moreover, if the cosmological constant is taken as zero $\Lambda = 0$, then we observed that the algebra among first and second class constraints form a Poincaré algebra, however, the algebra is still well defined by using the fact that the structure constants are those for the group $\text{SO}(2, 1)$; therefore, our results extend those reported in [5, 8, 9]. In fact, we could observe that it is not necessary to couple GR with matter fields in order to conclude that the Poisson algebra is closed provided that the internal group is $\text{SO}(2, 1)$, independently if matter fields and/or cosmological constant are present or not; in general our results indicate that the Poisson algebra is closed if the group of 3D GR written in the first order formalism corresponds to $\text{SO}(2, 1)$. We also compute Dirac's algebra among the constraints, we showed that the algebra is closed. Furthermore, in [9] it is concluded that the gauge symmetry of Palatini's theory without a cosmological constant correspond to Poincaré symmetry. In fact, by taking in our results $\Lambda = 0$, the complete structure of the first class constraints found in this work allowed to

construct a gauge generator, and we conclude that the gauge symmetries correspond to Poincaré transformations confirming the results reported in [9]. Finally, our results allowed us to construct the fundamental Dirac's brackets of the theory and determine the full set of Lagrange multipliers, we then constructed the extended action. Therefore, we conclude this work by pointing out that it is mandatory to perform a detailed Dirac's analysis in order to identify the correct symmetries of the theory under study.

We finish with some comments. We are able to observe that in Palatini's theory it is possible to take the cosmological constant as zero. In fact, the Dirac brackets (2.27) do not contain terms with the cosmological constant. This fact shows a difference among the so-called exotic action for gravity [8, 28, 29] and Palatini's theory. In fact, in [29] a detailed canonical analysis of exotic gravity is performed, and the following results are reported; there is not any restriction about the internal group, the gauge symmetry is an Λ -deformed Poincaré symmetry. However, the Dirac brackets among the dynamical variables are non-commutative and the cosmological constant can not vanish. In fact, the Dirac brackets for exotic action are given by [29]

$$\begin{aligned}
 \{e^I_a(x), e^J_b(y)\}_D &= \frac{1}{\Lambda} \eta^{IJ} \epsilon_{0ab} \delta^2(x-y), \\
 \{\Pi^a_I(x), \Pi^b_J(y)\}_D &= \frac{\Lambda}{4} \eta_{IJ} \epsilon^{0ab} \delta^2(x-y), \\
 \{A^{IJ}_a(x), \Pi^b_{LN}(y)\}_D &= \frac{1}{4} \delta^b_a [\delta^I_L \delta^J_N - \delta^I_N \delta^J_L] \delta^2(x-y), \\
 \{A^{IJ}_a(x), A^{LN}_b(y)\}_D &= \frac{1}{2} [\eta^{IL} \eta^{JN} - \eta^{IN} \eta^{JL}] \epsilon_{0ab} \delta^2(x-y), \\
 \{\Pi^a_{IJ}(x), \Pi^b_{LN}(y)\}_D &= \frac{1}{8} [\eta_{IL} \eta_{JN} - \eta_{IN} \eta_{JL}] \epsilon^{0ab} \delta^2(x-y).
 \end{aligned} \tag{3.1}$$

In this manner, we observe a difference at classical level among exotic action for gravity and Palatini's theory (see [29]). In fact, the Dirac brackets (2.27) are commutative and are different from (3.1); in (3.1) we can not take $\Lambda = 0$.

Finally, we would to comment that recently results confirming differences among exotic action and Palatini's gravity in the context of black holes have been published, in fact, we can observe in [30] results on exotic black hole giving some differences of exotic gravity from the normal gravity action.

A Algebra among the constraints

In this appendix we develop the algebra of the constraints $\{\gamma_I(x), \chi^a_J(y)\}$ and $\{\chi^a_I(x), \chi^b_{KL}(y)\}$ given by

$$\begin{aligned}
 \{\gamma_{IJ}(x), \chi^a_{KL}(y)\} &= \frac{1}{2} [\eta_{KJ} \Pi^a_{IL} - \eta_{KI} \Pi^a_{JL} + \eta_{LI} \Pi^a_{JK} - \eta_{LJ} \Pi^a_{IK} \\
 &\quad - \epsilon^{0ab} \epsilon^H_{IM} \eta_{JQ} \epsilon^{FMQ} \epsilon_{KLF} e_{bH} - \epsilon^{0ab} \epsilon_{MFJ} \eta_{IQ} \epsilon^{HQM} \epsilon_{KLH} e_b^F] \delta^2(x-y) \\
 &= \frac{1}{2} [\eta_{KJ} \Pi^a_{IL} - \eta_{KI} \Pi^a_{JL} + \eta_{LI} \Pi^a_{JK} - \eta_{LJ} \Pi^a_{IK} \\
 &\quad + \epsilon^{0ab} \epsilon^H_{IM} \eta_{JQ} (\delta^K_M \delta_L^Q - \delta_L^M \delta^K_Q) e_{bH}
 \end{aligned}$$

$$\begin{aligned}
 & + \epsilon^{0ab} \epsilon_{MFJ} \eta_{IQ} \left(\delta_K^Q \delta_L^M - \delta_L^Q \delta_K^M \right) e_b^F \Big] \delta^2(x-y) \\
 & = \frac{1}{2} \left[\eta_{KJ} \left(\Pi^a_{IL} - \epsilon^{0ab} \epsilon^H_{IL} e_{bH} \right) - \eta_{KI} \left(\Pi^a_{JL} - \epsilon^{0ab} \epsilon^H_{JL} e_{bH} \right) \right. \\
 & \quad \left. + \eta_{LI} \left(\Pi^a_{JK} - \epsilon^{0ab} \epsilon^H_{JK} e_{bH} \right) - \eta_{LJ} \left(\Pi^a_{IK} - \epsilon^{0ab} \epsilon^H_{IK} e_{bH} \right) \right] \delta^2(x-y) \\
 & = \frac{1}{2} \left[\eta_{KJ} \chi^a_{IL} - \eta_{KI} \chi^a_{JL} + \eta_{LI} \chi^a_{JK} - \eta_{LJ} \chi^a_{IK} \right] \delta^2(x-y) \approx 0, \tag{A.1} \\
 \{ \gamma_{IJ}(x), \gamma_{KL}(y) \} & = \frac{1}{2} \left[\eta_{KI} \left[D_a \Pi^a_{LJ} + \frac{1}{2} (\Pi^a_{LeaJ} - \Pi^a_{JeaL}) \right] \right. \\
 & \quad - \eta_{LI} \left[D_a \Pi^a_{KJ} + \frac{1}{2} (\Pi^a_{KeaJ} - \Pi^a_{JeaK}) \right] \\
 & \quad + \eta_{KJ} \left[D_a \Pi^a_{IL} + \frac{1}{2} (\Pi^a_{IeaL} - \Pi^a_{LeaI}) \right] \\
 & \quad \left. - \eta_{JL} \left[D_a \Pi^a_{IK} + \frac{1}{2} (\Pi^a_{IeaK} - \Pi^a_{KeaI}) \right] \right] \delta^2(x-y) \\
 & = \frac{1}{2} \left[\eta_{KI} \gamma_{JL} - \eta_{LJ} \gamma_{KI} + \eta_{KJ} \gamma_{IL} - \eta_{JL} \gamma_{IK} \right] \delta^2(x-y) \approx 0. \tag{A.2}
 \end{aligned}$$

where we have used (2.25) and the following expression in order to obtain a closed algebra

$$\begin{aligned}
 \frac{1}{2} \epsilon^H_{IM} \epsilon^{MFJ} (\Pi^a_{FeaH} - \Pi^a_{HeaF}) & = \frac{1}{2} \epsilon_{HIM} \epsilon^{MFQ} \eta_{QJ} (\Pi^a_{Fea}{}^H - \Pi^a{}^H e_{aF}) \\
 & = \frac{1}{2} (-1) \left(\delta^F_H \delta^Q_I - \delta^F_I \delta^Q_H \right) \eta_{QJ} (\Pi^a_{Fea}{}^H - \Pi^a{}^H e_{aF}) \\
 & = \frac{1}{2} (\Pi^a_{IeaJ} - \Pi^a_{JeaI}). \tag{A.3}
 \end{aligned}$$

Finally we compute the Poisson bracket among $\{\gamma_I(x), \gamma_{MN}(y)\}$,

$$\begin{aligned}
 \{ \gamma_I(x), \gamma_{MN}(y) \} & = \{ -D_a \chi^a_I, D_a \Pi^a_{MN} + \frac{1}{2} \epsilon^H_{MQ} \epsilon^{QF}_N (\chi^a_{FeaH} - \chi^a_{HeaF}) \} \\
 & \quad + \{ -\frac{1}{2} \epsilon^{0ab} \epsilon_{IKL} F_{ab}{}^{KL}, D_a \Pi^a_{MN} + \frac{1}{2} \epsilon^H_{MQ} \epsilon^{QF}_N (\chi^a_{FeaH} - \chi^a_{HeaF}) \} \\
 & \quad + \{ \Lambda e_a{}^J \Pi^a_{IJ} + \Lambda e_a{}^J \chi^a_{IJ}, D_a \Pi^a_{MN} + \frac{1}{2} \epsilon^H_{MQ} \epsilon^{QF}_N (\chi^a_{FeaH} - \chi^a_{HeaF}) \}, \tag{A.4}
 \end{aligned}$$

first, we calculate the following bracket

$$\begin{aligned}
 & \star \{ -D_a \chi^a_I, D_a \Pi^a_{MN} + \frac{1}{2} \epsilon^H_{MQ} \epsilon^{QF}_N (\chi^a_{FeaH} - \chi^a_{HeaF}) \} \\
 & = -\frac{1}{2} \epsilon^H_{MQ} \epsilon^{QF}_N \{ D_a \Pi^a_I, \Pi^a_{FeaH} - \Pi^a_{HeaF} \} \\
 & = -\frac{1}{2} \epsilon^H_{MQ} \epsilon^{QF}_N [\eta_{IF} D_a \Pi^a_H - \eta_{IH} D_a \Pi^a_F] \delta^2(x-y) \\
 & = \frac{1}{2} [\eta_{NI} D_a \chi^a_M - \eta_{MI} D_a \chi^a_N] \delta^2(x-y), \tag{A.5}
 \end{aligned}$$

where we have used (2.25).

$$\begin{aligned}
 & \star \left\{ -\frac{1}{2} \epsilon^{0ab} \epsilon_{IKL} F_{ab}{}^{KL}, D_a \Pi^a{}_{MN} + \frac{1}{2} \epsilon^H{}_{MQ} \epsilon^{QF}{}_N (\chi^a{}_F e_{aH} - \chi^a{}_H e_{aF}) \right\} \\
 &= \left\{ -\frac{1}{2} \epsilon^{0ab} \epsilon_{IKL} F_{ab}{}^{KL}, D_a \Pi^a{}_{MN} \right\} \\
 &= \frac{1}{2} \left[-\frac{1}{2} \epsilon^{0ab} \eta_{MI} \epsilon_{NEL} F_{ab}{}^{EL} + \frac{1}{2} \epsilon^{0ab} \eta_{NI} \epsilon_{MEL} F_{ab}{}^{EL} \right] \delta^2(x-y).
 \end{aligned} \tag{A.6}$$

Furthermore, we calculate the following bracket

$$\begin{aligned}
 & \star \{ \Lambda e_a{}^J \Pi^a{}_{IJ} + \Lambda e_a{}^J \chi^a{}_{IJ}, D_a \Pi^a{}_{MN} + \frac{1}{2} \epsilon^H{}_{MQ} \epsilon^{QF}{}_N (\chi^a{}_F e_{aH} - \chi^a{}_H e_{aF}) \} \\
 &= \frac{\Lambda}{2} [\eta_{MI} e_a{}^F \Pi^a{}_{NF} - \eta_{NI} e_a{}^E \Pi^a{}_{ME} + \eta_{MI} e_a{}^F \chi^a{}_{NF} - \eta_{NI} e_a{}^E \chi^a{}_{ME}] \delta^2(x-y),
 \end{aligned} \tag{A.7}$$

where also we have used (2.25). Therefore we obtain

$$\{ \gamma_I(x), \gamma_{MN}(y) \} = \frac{1}{2} [\eta_{IM} \gamma_N - \eta_{IN} \gamma_M] \delta^2(x-y) \approx 0,$$

this shows that the Poisson algebra is closed provided the structure constants correspond to the group $SO(2, 1)$.

B The Dirac brackets among the constraints

By using the brackets given in (2.27), we calculate the Dirac brackets among the first class and second class constraints given by

$$\begin{aligned}
 \{ \gamma_I(x), \gamma_J(y) \}_D &= 2\Lambda \left[\gamma_{JI} + \frac{1}{2} \epsilon_{0ab} (\epsilon^M{}_J{}^L \chi^a{}_{IM} \chi^b{}_L - \epsilon^M{}_I{}^L \chi^a{}_{JM} \chi^b{}_L) \right] \delta^2(x-y) \approx 0, \\
 \{ \gamma_I(x), \gamma_{MN}(y) \}_D &= \frac{1}{2} \left[\eta_{IM} \gamma_N - \eta_{IN} \gamma_M + \frac{1}{2} \epsilon_{0ab} (\epsilon_I{}^E{}_M \chi^a{}_N \chi^b{}_E - \epsilon_I{}^E{}_N \chi^a{}_M \chi^b{}_E) \right. \\
 &\quad \left. + 2\Lambda \epsilon_{0ab} (\epsilon^{KE}{}_N \chi^a{}_{IK} \chi^b{}_{ME} - \epsilon^{KE}{}_M \chi^a{}_{IK} \chi^b{}_{NE}) \right] \delta^2(x-y) \approx 0, \\
 \{ \gamma_{IJ}(x), \gamma_{MN}(y) \}_D &= \frac{1}{2} [\eta_{IM} \gamma_{NJ} - \eta_{IN} \gamma_{MJ} + \eta_{JM} \gamma_{IN} - \eta_{JN} \gamma_{IM} \\
 &\quad + \frac{1}{2} \epsilon_{0ab} \epsilon_I{}^D{}_M (\chi^a{}_J \chi^b{}_{ND} + \chi^a{}_N \chi^b{}_{JD}) \\
 &\quad + \frac{1}{2} \epsilon_{0ab} \epsilon_I{}^D{}_N (\chi^a{}_J \chi^b{}_{DM} + \chi^a{}_M \chi^b{}_{DJ}) \\
 &\quad + \frac{1}{2} \epsilon_{0ab} \epsilon_J{}^D{}_M (\chi^a{}_I \chi^b{}_{DN} + \chi^a{}_N \chi^b{}_{DI}) \\
 &\quad + \frac{1}{2} \epsilon_{0ab} \epsilon_J{}^D{}_N (\chi^a{}_I \chi^b{}_{MD} + \chi^a{}_M \chi^b{}_{ID})] \delta^2(x-y) \approx 0, \\
 \{ \gamma_I(x), \chi^a{}_J(y) \}_D &= 0, \\
 \{ \gamma_I(x), \chi^a{}_{KL}(y) \}_D &= 0, \\
 \{ \gamma_{IJ}(x), \chi^a{}_K(y) \}_D &= 0, \\
 \{ \gamma_{IJ}(x), \chi^a{}_{KL}(y) \}_D &= 0, \\
 \{ \chi^a{}_I(x), \chi^b{}_{KL}(y) \}_D &= 0,
 \end{aligned} \tag{B.1}$$

hence the algebra is closed. We can observe that only squares of second class constraints appear. In fact, the Dirac brackets among first class constraints must be square of second class constraints and linear of first class constraints [14], thus, this calculation shows that our results has a form desired. In addition, we have used the equation (2.25) in order to obtain that algebra. It is important to comment that we have showed that Dirac's procedure work with the internal group $SO(2,1)$ and the corresponding algebra for other internal general groups is not yet solved, however, we are working on this subject and the results will be reported in forthcoming works.

C Comments on standard Dirac's method

We have proved that the Poisson and Dirac's algebra is closed, the relation given by $\epsilon^{IJK}\epsilon_{IMN} = (-1)(\delta^J_M\delta^K_N - \delta^J_N\delta^K_M)$ is a property of the structure constants $\epsilon^I_{JK} = \epsilon^{IMN}\eta_{MJ}\eta_{NK}$ of the Lie algebra of $SO(2,1)$. However, we can observe in [5] an analysis for Palatini theory with $\Lambda \neq 0$ performed by using the adjoint representation of the internal group \mathcal{G} , and the Hamiltonian analysis was developed on a smaller phase space context. In fact, in [5] it is proved that Palatini theory (with or without a cosmological constant Λ) is well-defined for a wide class of Lie groups. If $\Lambda = 0$, the Lie group \mathcal{G} can be completely arbitrary; if $\Lambda \neq 0$, then \mathcal{G} has to admit an invariant totally anti-symmetric tensor ϵ^{IJK} . However, in order to couple degrees of freedom to 3D gravity, for instance a scalar field, the algebra among the first class constraints is closed for the internal group $SO(2,1)$. On the other hand, the goal of our paper is that without work with the adjoint representation of the Lie algebra of \mathcal{G} and by using a pure Dirac's analysis, we have showed that the Poisson's algebra of first class constraints is closed provided that the internal group is $SO(2,1)$, and it is not necessary to couple matter fields to 3D gravity in order to obtain those conclusions. Therefore, if we work with the adjoint representation of the Lie algebra of $SO(2,1)$, then 3D Palatini theory with $\Lambda \neq 0$ is well-defined for semi-simple Lie groups, the algebra of the constraints is closed but does not form a Poincaré algebra. Let us show this point; by using the adjoint representation of the Lie algebra of $SO(2,1)$, the action (2.1) takes the following form

$$S[e, A] = \int_M R[A]^I \wedge e_I - \frac{\Lambda}{3} \epsilon^{IJK} e_I \wedge e_J \wedge e_K. \quad (C.1)$$

Hence, by performing the Hamiltonian analysis, we find the following first class constraints

$$\begin{aligned} \gamma_I &= D_a \Pi_I^a, \\ \Gamma^I &= \epsilon^{0ab} F_{ab}^I - \Lambda \epsilon_{0ab} \epsilon^{IJK} \Pi_J^a \Pi_K^a, \end{aligned} \quad (C.2)$$

where (Π_I^a, π_K^a) are the momenta canonically conjugated to (A_a^I, e_a^I) , and $D_a \lambda^I = \partial_a \lambda^I + \epsilon^I_{JK} A^J \lambda^K$. The algebra among the first class constraints is given by

$$\begin{aligned} [\gamma_I, \gamma_J] &= \epsilon_{IJ}^K \gamma_K, \\ [\gamma_I, \Gamma^J] &= \epsilon_I^{JK} \Gamma_K, \\ [\Gamma_I, \Gamma_J] &= \Lambda \epsilon_{IJ}^K \Gamma_K. \end{aligned} \quad (C.3)$$

In this manner, the algebra among the constraints is closed, however, in order to obtain that algebra it is not necessary to restrict ourselves to $\text{SO}(2,1)$. In fact, by using the adjoint representation of the Lie group \mathcal{G} the algebra (C.3) reproduces the results reported in [5], namely, 3D gravity with a cosmological constant is well-defined for semi-simple Lie groups and the algebra (C.3) is closed but does not form a Poincaré algebra. On the other hand, by working without the adjoint representation as is done in our work, the Poisson algebra among the constraints is closed provided that $\mathcal{G} = \text{SO}(2,1)$ and the algebra among the constraints form an Λ -deformed Poincaré algebra.

D Poincaré transformations

The gauge symmetries obtained in (2.34), are related with Poincaré transformations. In fact, let us study the case when the cosmological constant vanishes; by considering the following Lie-algebra valued one-form

$$\omega_\mu = e_\mu^I P_I + \frac{1}{2} A_\mu^{IK} M_{IK}, \quad (\text{D.1})$$

where P_I and M_{IK} are the Poncaré generators. By writing $M^I = \frac{1}{2} \epsilon^{IKL} M_{KL}$, the generators obey the standard commutation relations

$$\begin{aligned} [M^I, M^K] &= \epsilon^{IKL} M_L, \\ [M^I, P^K] &= \epsilon^{IKL} P_L, \\ [P^I, P^K] &= 0, \end{aligned} \quad (\text{D.2})$$

here $I, J, K = 0, 1, 2$.

By considering the variation of ω under the gauge symmetry of kind $\delta\omega = D\lambda = \partial\lambda + [\omega, \lambda]$ where $\lambda = \lambda^I P_I + \frac{1}{2} \lambda^{IK} M_{IK}$, we obtain the following components [25]

Translations

$$\begin{aligned} e_\mu^I &\longrightarrow e_\mu^I + D_\mu \lambda^I, \\ A_\mu^{IJ} &\longrightarrow A_\mu^{IJ}. \end{aligned} \quad (\text{D.3})$$

Lorentz transformations (rotations)

$$\begin{aligned} e_\mu^I &\longrightarrow e_\mu^I + \lambda^{IJ} e_{J\mu}, \\ A_\mu^{IJ} &\longrightarrow A_\mu^{IJ} + D_\mu \lambda^{IJ}. \end{aligned} \quad (\text{D.4})$$

We can see that these transformations are those obtained in (2.34) with $\Lambda = 0$.

It is possible generalise the above results for $\Lambda \neq 0$. In fact, now the generators obey the following algebra [25]

$$\begin{aligned} [M^I, M^K] &= \epsilon^{IKL} M_L, \\ [M^I, P^K] &= \epsilon^{IKL} P_L, \\ [P^I, P^K] &= \Lambda \epsilon^{IKL} M_L. \end{aligned} \quad (\text{D.5})$$

By considering this algebra, we obtain the transformation laws found in eq. (2.34). However, in our work the transformations (2.34) were obtained by using a pure Dirac's method.

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